CS 1671/2071
Human Language Technologies

Session 11: Logistic regression, part 2

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University of Pittsburgh | School of Computing and Information

Course logistics

- Homework 2 is due **this Thu Feb 20**
- If I emailed your group about choosing different directions or datasets and I haven't heard from you, I'll check in with you this week
- Next project milestone: project proposal due Feb 28
    - I will release instructions for that early this week

Midterm course evaluation (OMETs)

- CS 1671: https://go.blueja.io/BJVNkUaUE0WIdL6VHILkXQ
- CS 2071: https://go.blueja.io/fiEDPP0eM0eQ3kzYBucv6w
- All types of feedback are welcome (critical and positive)
- Completely anonymous, will not affect grades
- Let me know what’s working and what to improve on while the course is still running!
- Please be as specific as possible
- Available until **Wed Feb 19**

Lecture overview: logistic regression part 2

- **Learning the weights for features in logistic regression**
    - Cross-entropy loss function
    - Stochastic gradient descent
    - Batch and mini-batch training
- Coding activity: error analysis

Review: classification with logistic regression

1. What is the necessary format for the input to logistic regression? What will the output format be?

2. What is the equation for calculating $\hat{y}$, the predicted class from an input vector x?

Logistic regression: learning the weights

Wait, where did the *w*'s come from?

Supervised classification:

- We know the correct label y (either 0 or 1) for each x.
- But what the system produces is an estimate, $\hat{y}$

We want to set w and b to minimize the **distance** between our estimate $\hat{y}^{(i)}$ and the true $y^{(i)}$.

- We need a distance estimator: a **loss function** or a **cost function**
- We need an optimization algorithm to update w and b to minimize the loss.

Slide credit: Jurafsky & Martin

Learning components

A loss function:

cross-entropy loss

An optimization algorithm:

stochastic gradient descent

Slide credit: Jurafsky & Martin

The distance between $\hat{y}$ and y

We want to know how far is the classifier output:

$$\hat{y} = \sigma(w \cdot x + b)$$

from the true output:

$$y \qquad [= \text{either } 0 \text{ or } 1]$$

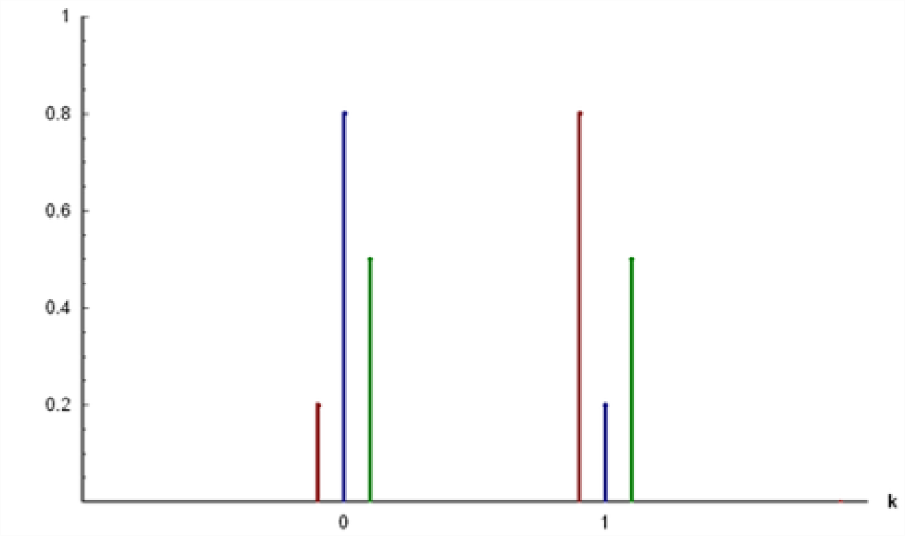
We'll call this difference:

$L(\hat{y}, y)$ = how much $\hat{y}$ differs from the true y

Slide credit: Jurafsky & Martin

Cross-entropy loss for binary classification

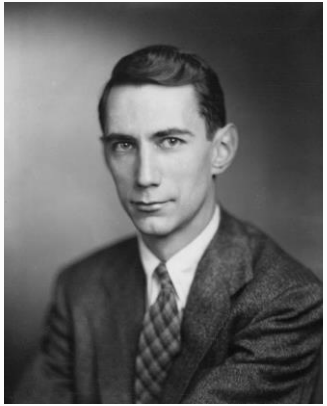
- Cross-entropy loss: measure of distance between true distribution and predicted probability distribution of labels
- Logistic regression predicts $p(y=0)$ and $p(y=1)$ in a Bernoulli distribution. The true labels can also be considered a Bernoulli distribution over possible labels. If y=1, p(y=1) = 1 and p(y=0) = 0.



Cross-entropy loss for binary classification

Cross-entropy between Bernoulli distributions of the predicted, where $\hat{y}$ is the predicted label and y is the true label

Minimize: $$L_{CE}(\hat{y}, y) = -[y \log \hat{y} + (1-y) \log(1-\hat{y})]$$



Claude Shannon

Slide credit: Jurafsky & Martin

Let's see if this works for our sentiment example

We want loss to be:

- smaller if the model estimate is close to correct
- bigger if model is confused

Let's first suppose the true label of this is y=1 (positive)

It's hokey . There are virtually no surprises , and the writing is second-rate . So why was it so enjoyable ? For one thing , the cast is great . Another nice touch is the music . I was overcome with the urge to get off the couch and start dancing . It sucked me in , and it'll do the same to you .

Slide credit: Jurafsky & Martin

Let's see if this works for our sentiment example

True value is y=1. How well is our model doing?

$$P(+|X) = P(Y = 1|X)$$
$$= \sigma(w \cdot \mathrm{x} + b) = \sigma(\textstyle\sum_{i=1}^{n} w_i x_i + b)$$
$$= \sigma((2.5*3)+ (-5.0*2)+ (-1.2*1)+ (0.5*3)+ (2.0*0)+ (0.7*4.19)+b)$$
$$= \sigma(0.733+0.1)$$
$$= \sigma(0.833) = 0.7$$

Pretty well! What's the loss?

$$\begin{aligned} L_{\mathrm{CE}}(\hat{y}, y) &= -[y \log \sigma(w \cdot x + b) + (1-y) \log(1 - \sigma(w \cdot x + b))] \\ &= -[\log \sigma(w \cdot x + b)] \\ &= -\log(.70) \\ &= .36 \end{aligned}$$

Slide credit: Jurafsky & Martin

Let's see if this works for our sentiment example

Suppose true value instead was y=0.

$p(y=0|x) = 1 - p(y=1|x)$
$= 1 - 0.7$
$= 0.3$

What's the loss?

$$
\begin{aligned}
L_{\mathrm{CE}}(\hat{y}, y) &= -[y \log \sigma(w \cdot x + b) + (1-y) \log(1 - \sigma(w \cdot x + b))] \\
&= -[\log(1 - \sigma(w \cdot x + b))] \\
&= -\log(.30) \\
&= 1.2
\end{aligned}
$$

Slide credit: Jurafsky & Martin

Let's see if this works for our sentiment example

The loss when model was right (if true y=1)

$$
\begin{aligned}
L_{\mathrm{CE}}(\hat{y},y) &= -[y\log\sigma(w\cdot x+b)+(1-y)\log(1-\sigma(w\cdot x+b))] \\
&= -[\log\sigma(w\cdot x+b)] \\
&= -\log(.70) \\
&= .36
\end{aligned}
$$

Is lower than the loss when model was wrong (if true y=0):

$$
\begin{aligned}
L_{\mathrm{CE}}(\hat{y},y) &= -[y\log\sigma(w\cdot x+b)+(1-y)\log(1-\sigma(w\cdot x+b))] \\
&= -[\log(1-\sigma(w\cdot x+b))] \\
&= -\log(.30) \\
&= 1.2
\end{aligned}
$$

Sure enough, loss was bigger in the case where the model was wrong!

Slide credit: Jurafsky & Martin

Stochastic gradient descent

Our Goal: Minimize the Loss

Let's make it explicit that the loss function is parameterized by weights $\theta = (w, b)$.

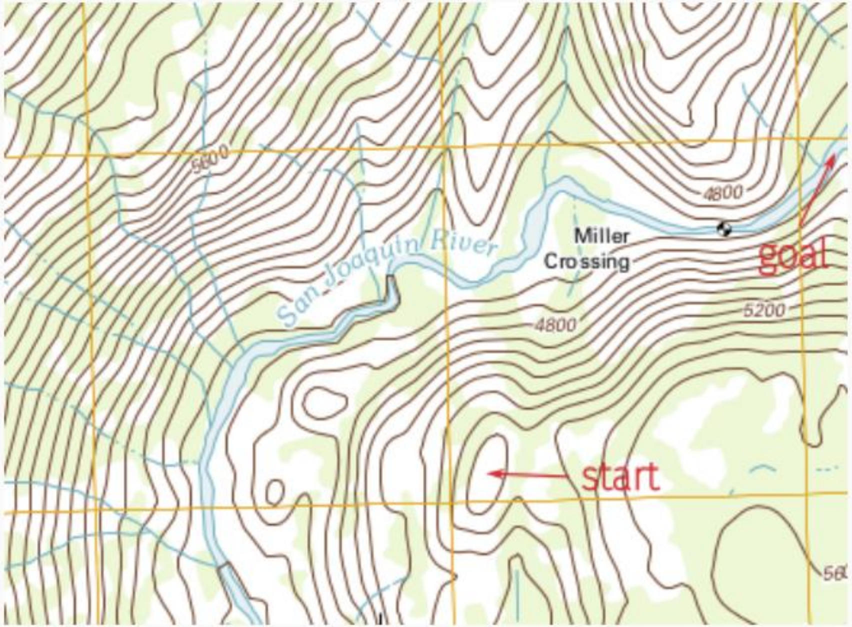
We'll represent $\hat{y}$ as $f(x; \theta)$ to make the dependency on θ more obvious.

We want the weights that minimize the loss (L_{CE}), averaged over all examples:

$$\hat{\theta} = \underset{\theta}{\operatorname{argmin}} \frac{1}{m} \sum_{i=1}^{m} L_{CE}(f(x^{(i)}; \theta), y^{(i)})$$

Slide credit: David Mortensen

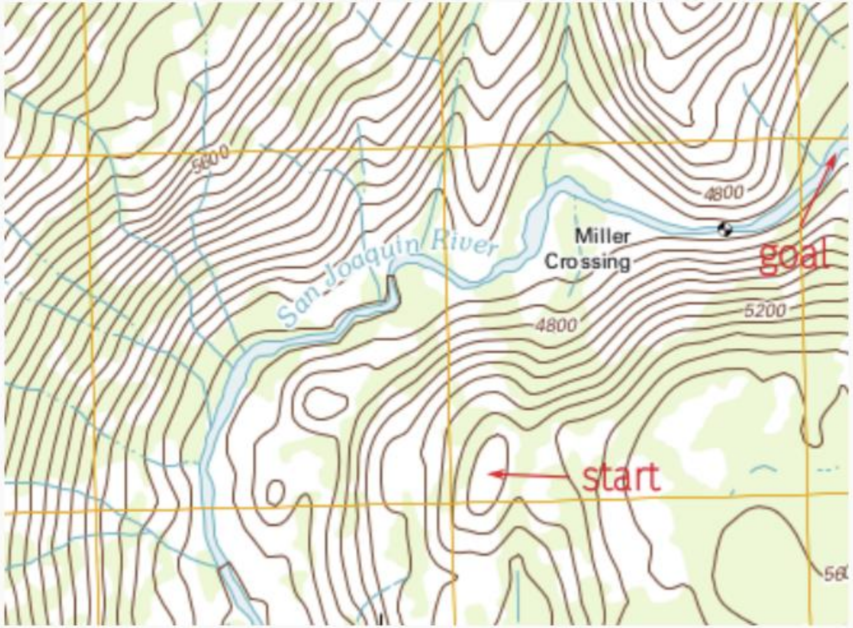
The Intuition of Gradient Descent



- You are on a hill
- It is your mission to reach the river at the bottom of the canyon (as quickly as possible)
- What is your strategy?

Slide credit: David Mortensen

The Intuition of Gradient Descent



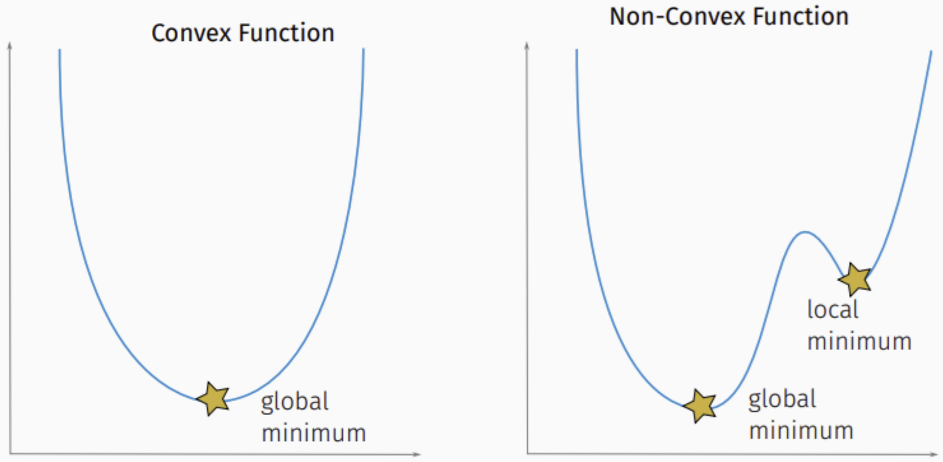
- You are on a hill
- It is your mission to reach the river at the bottom of the canyon (as quickly as possible)
- What is your strategy?
  1. Determine in which direction the steepest downhill slope lies
  2. Take a step in that direction
  3. Repeat until a step in any direction will take you up hill

Slide credit: David Mortensen

Our Goal: Minimize the Loss

For logistic regression, the loss function is **convex**

- Just one minimum
- Gradient descent is guaranteed to find the minimum, no matter where you start

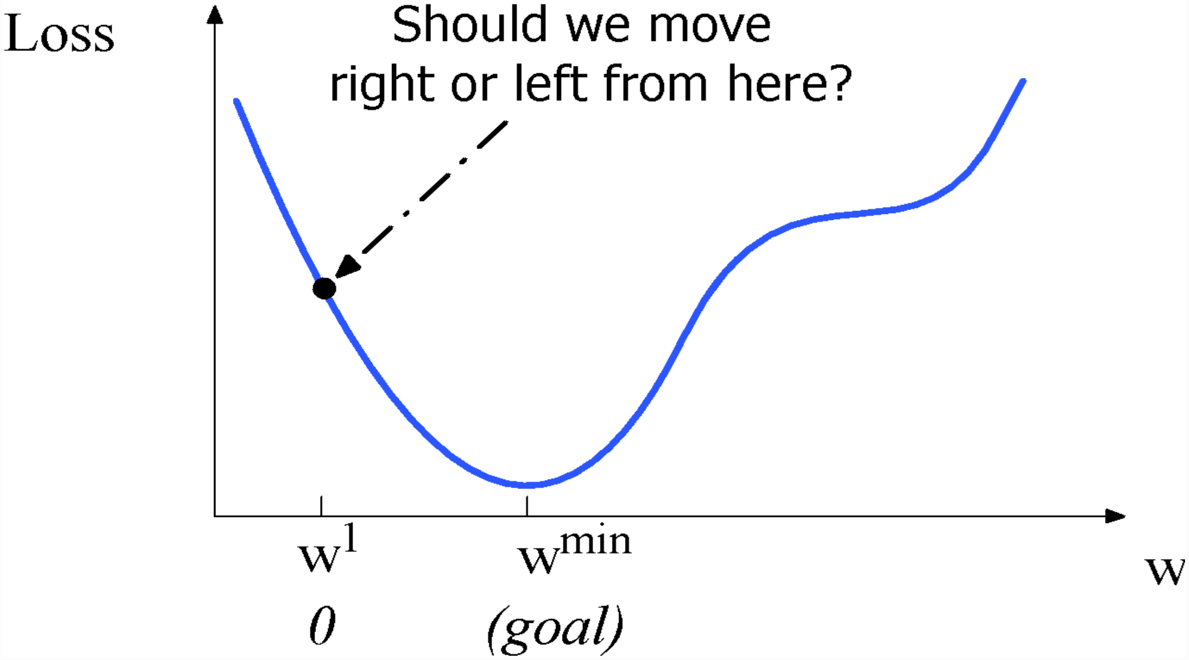


Slide credit: David Mortensen

Let's first visualize for a single scalar w

Q: Given current w, should we make it bigger or smaller?

A: Move *w* in the reverse direction from the slope of the function



Slide adapted from Jurafksy & Martin

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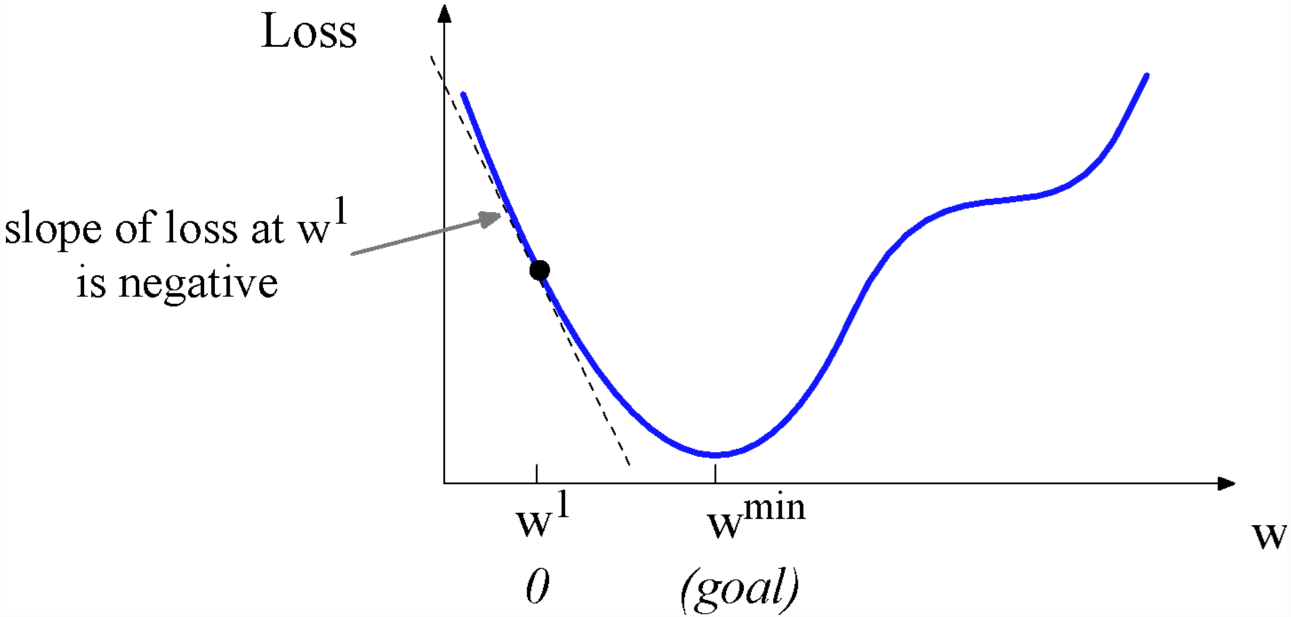

So we'll move
positive (to the right)

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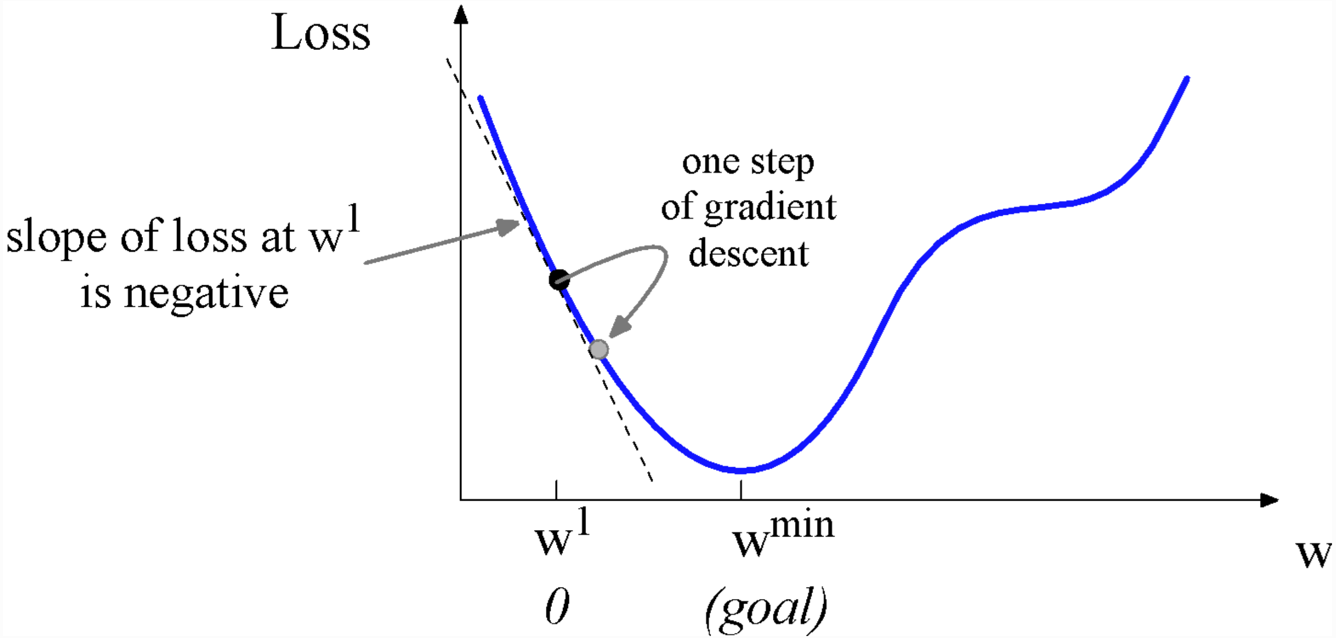

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Slide adapted from Jurafksy & Martin

A Gradient is a Vector Pointing in the Direction of Greatest Increase

The GRADIENT of a function of many variables is a vector pointing in the direction of the greatest increase in a function.

GRADIENT DESCENT: Find the gradient of the loss function at the current point and move in the **opposite** direction.

Slide credit: David Mortensen

How Much Do We Move in a Step?

- We move by the value of the gradient (in our example, the slope)

$$\frac{d}{d\mathbf{w}} L_{CE}(f(\mathbf{x};\mathbf{w}), y)$$

  weighted by the LEARNING RATE η
- The higher the learning rate, the faster $\mathbf{w}$ changes:

$$\mathbf{w}_{t+1} = \mathbf{w}_t - \eta \frac{d}{dw} L_{CE}(f(\mathbf{x};\mathbf{w}), y)$$

Slide credit: David Mortensen

How Do We Do Gradient Descent in N Dimensions?

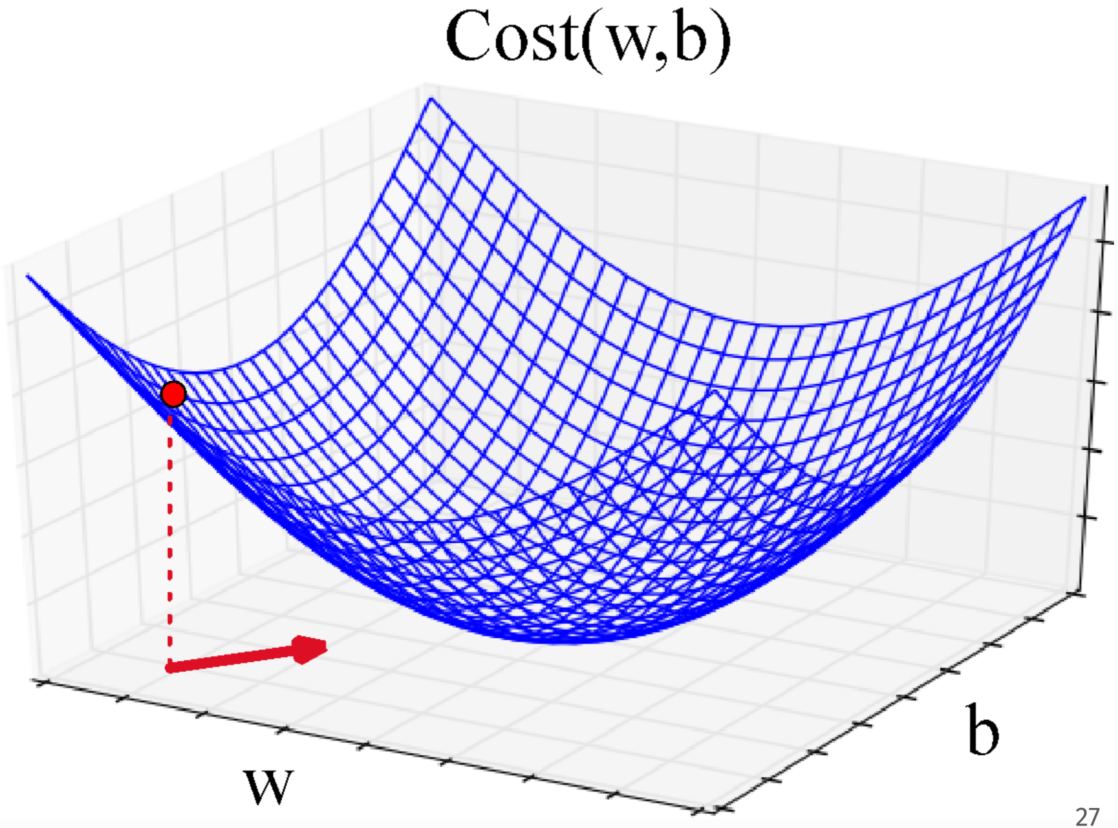
We want to know where in the N-dimensional space (of the N parameters that make up θ) we should move.

The **gradient is just such a vector**; it expresses the directional components of the sharpest slope along each of the N dimensions.

Imagine 2 dimensions, *w* and *b*

Visualizing the gradient vector at the red point

It has two dimensions shown in the x-y plane



But Real Gradients Have More than Two Dimensions

- They are much longer
- They have lots of weights
- For each dimension w_i, the gradient component i tells us the slope w.r.t. that variable
  - "How much would a small change in w_i influence the total loss function L?"
  - The slope is expressed as the partial derivative ∂ of the loss ∂w_i
- We can then define the gradient as **a vector of these partials**

Slide credit: David Mortensen

Computing the Gradient

Let's represent $\hat{y}$ as $f(x;\theta)$ to make things clearer:

$$\nabla_\theta L(f(\mathbf{x};\theta),y) = \begin{bmatrix} \frac{\partial}{\partial w_0} L(f(\mathbf{x};\theta),y) \\ \frac{\partial}{\partial w_1} L(f(\mathbf{x};\theta),y) \\ \frac{\partial}{\partial w_2} L(f(\mathbf{x};\theta),y) \\ \vdots \\ \frac{\partial}{\partial w_n} L(f(\mathbf{x};\theta),y) \end{bmatrix}$$

Note that, since we are representing the bias b as w_0, θ is more-or-less equivalent to $\mathbf{w}$.

What is the final equation for updating θ based on the gradient?

$$\theta_{t+1} = \theta_t - \eta \nabla L(f(x;\theta),y)$$

(For us, L is the cross-entropy loss L_{CE}).

Slide credit: David Mortensen

So What Are These Partial Derivatives Used in Logistic Regression?

The textbook lays out the derivation in §5.10 but here's the basic idea:

Here is the cross-entropy loss function (for binary classification):

$$L_{CE}(\hat{y}, y) = -[y \log \sigma(w \cdot x + b) + (1 - y) \log(1 - \sigma(w \cdot x + b))]$$

The derivative of this function is:

$$\frac{\partial L_{CE}(\hat{y}, y)}{\partial w_j} = [\sigma(w \cdot x + b) - y]x_j$$

which is very manageable!

Slide credit: David Mortensen

function STOCHASTIC GRADIENT DESCENT($L()$, $f()$, x, y) **returns** θ

\# where: L is the loss function

\# f is a function parameterized by θ

\# x is the set of training inputs $x^{(1)}, x^{(2)}, \ldots, x^{(m)}$

\# y is the set of training outputs (labels) $y^{(1)}, y^{(2)}, \ldots, y^{(m)}$

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repeat til done   # see caption

For each training tuple $(x^{(i)}, y^{(i)})$ (in random order)

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  For each training tuple $(x^{(i)}, y^{(i)})$ (in random order)

    1. Optional (for reporting): # How are we doing on this tuple?

    Compute $\hat{y}^{(i)} = f(x^{(i)}; \theta)$ # What is our estimated output $\hat{y}$?

    Compute the loss $L(\hat{y}^{(i)}, y^{(i)})$ # How far off is $\hat{y}^{(i)}$ from the true output $y^{(i)}$?

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    # where: L is the loss function
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    2. $g \leftarrow \nabla_\theta L(f(x^{(i)};\theta), y^{(i)})$ # How should we move θ to maximize loss?

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2. $g \leftarrow \nabla_\theta L(f(x^{(i)};\theta), y^{(i)})$ # How should we move θ to maximize loss?
3. $\theta \leftarrow \theta - \eta\, g$ # Go the other way instead

return θ

Slide adapted from Jurafksy & Martin

A Sidenote: Hyperparameters

The learning rate (our η) is a **hyperparameter**, a term you will keep hearing

- **Set it too high?** The learner will catapult itself across the minimum and may not converge
- **Set it too low?** The learner will take a long time to get to the minimum, and may not converge in our lifetime

But what are hyperparameters again?

- Hyperparameters are parameters in a machine learning model that are not learned empirically
- They have to be set by the human who is designing the algorithm

Slide credit: David Mortensen

Working through an example

One step of gradient descent

A mini-sentiment example, where the true y=1 (positive)

Two features:

$x_1 = 3$ (count of positive lexicon words)

$x_2 = 2$ (count of negative lexicon words)

Assume 3 parameters (2 weights and 1 bias) in Θ^0 are zero:

$w_1 = w_2 = b = 0$

$\eta = 0.1$

Slide adapted from Jurafksy & Martin

Example of gradient descent

Update step for update θ is:

$$\theta_{t+1} = \theta_t - \eta \frac{d}{d\theta} L(f(x;\theta),\, y)$$

$w_1 = w_2 = b = 0;$
$x_1 = 3; \quad x_2 = 2;$
$y = 1$

where $\quad \frac{\partial L_{\mathrm{CE}}(\hat{y},y)}{\partial w_j} = [\sigma(w\cdot x+b)-y]x_j$

Gradient vector has 3 dimensions:

$$\nabla_{w,b} = \begin{bmatrix} \frac{\partial L_{\mathrm{CE}}(\hat{y},y)}{\partial w_1} \\ \frac{\partial L_{\mathrm{CE}}(\hat{y},y)}{\partial w_2} \\ \frac{\partial L_{\mathrm{CE}}(\hat{y},y)}{\partial b} \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}$$

Slide adapted from Jurafksy & Martin

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Slide adapted from Jurafksy & Martin

Example of gradient descent

Now that we have a gradient, we compute the new parameter vector θ^1 by moving θ^0 in the opposite direction from the gradient:

$$\theta_{t+1} = \theta_t - \eta \frac{d}{d\theta} L(f(x;\theta),\, y)$$

$\eta = 0.1$

$$\theta^1 =$$

Example of gradient descent

Now that we have a gradient, we compute the new parameter vector θ^1 by moving θ^0 in the opposite direction from the gradient:

$$\theta_{t+1} = \theta_t - \eta \frac{d}{d\theta} L(f(x;\theta),\ y)$$

$\eta = 0.1$

$$\theta^1 = \begin{bmatrix} w_1 \\ w_2 \\ b \end{bmatrix} - \eta \begin{bmatrix} -1.5 \\ -1.0 \\ -0.5 \end{bmatrix} =$$

Batch and mini-batch training

Mini-batching

- In stochastic gradient descent, the algorithm chooses one random example at each iteration
- The result? Sometimes movements are choppy and abrupt
- In practice, instead, we usually compute the gradient over **batches** of training instances
- Entire dataset: BATCH TRAINING
- m examples (e.g., 512 or 1024): MINI-BATCH TRAINING

Slide credit: David Mortensen

Coding activity

Notebook: custom features for logistic regression

- Click on this nbgitpuller link
  - Or find the link on the course website
- Open **session11_error_analysis.ipynb**